

## A new set of equations describing immiscible two-phase flow in isotropic porous media

Alex Hansen · Santanu Sinha · Dick Bedeaux · Signe Kjelstrup · Isha Savani · Morten Vassvik

Received: date / Accepted: date

**Abstract** Based on non-equilibrium thermodynamics we derive a set of general equations relating the partial volumetric flow rates to each other and to the total volumetric flow rate in immiscible two-phase flow in porous media. These equations together with the conservation of saturation reduces the immiscible two-phase flow problem to a single-phase flow problem of a complex fluid. We discuss the new equation in terms of the relative permeability equations. We test the equations on model systems, both analytically and numerically.

**Keywords** Immiscible two-phase flow · Gibbs-Duhem relation · non-equilibrium thermodynamics · relative permeability · capillary pressure

---

Alex Hansen · Santanu Sinha · Isha Savani · Morten Vassvik  
Department of Physics  
Norwegian University of Science and Technology  
NO-7491 Trondheim  
Norway  
Tel.: +47-73593649  
E-mail: Alex.Hansen@ntnu.no  
*Present address of Santanu Sinha:* CSRC  
10 West Dongbeiwang Road  
Haidan District, Beijing 100193  
China

Dick Bedeaux · Signe Kjelstrup  
Department of Chemistry  
Norwegian University of Science and Technology  
NO-7491 Trondheim  
Norway  
Tel.: +47-73594179  
E-mail: Signe.Kjelstrup@ntnu.no

## 1 Introduction

The simultaneous flow of immiscible fluids through porous media has been studied for a long time [1]. It is a problem that lies at the heart of many important geophysical and industrial processes. Often, the length scales in the problem span numerous decades; from the pores measured in micrometers to reservoir scales measured in kilometers. At the largest scales, the porous medium is treated as a continuum governed by effective equations that encode the physics at the pore scale.

The problem of tying the pore scale physics together with the effective description at large scale is the upscaling problem. In 1936, Wycoff and Botset proposed a generalization of the Darcy equation to immiscible two-phase flow [2]. It is instructive to reread Wycoff and Botset's article. This is where the concept of relative permeability is introduced. The paper is eighty years old and yet it is still remarkably modern. Capillary pressure was first considered by Richards as early as 1931 [3]. In 1940, Leverett combined capillary pressure with the concept of relative permeability, and the framework describing essentially all later analysis of immiscible multiphase flow in porous media was in place [4].

The introduction of the concepts of relative permeability and capillary pressure as solution to the upscaling problem, dominates still today but other theories exist [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. These theories are, as is relative permeability, based on a number of detailed assumptions concerning the porous medium and concerning the physics involved.

It is the aim of this paper to present a new theory for flow in porous media that is solely based on a thermodynamic balance between work done on the flowing fluids and the dissipation in them. In the same way as Buckley and Leverett's analysis based on the conservation of the mass of the fluids in the porous medium led to their Buckley-Leverett equation [20], the balancing of work per time against dissipation leads to the new equations we present in the following.

The theory we present rests on non-equilibrium thermodynamics [21, 22, 23] which combines conservation laws with the laws of thermodynamics. The structure of the theory is reminiscent of the structure of thermodynamics itself: we have a number of variables that are related through general thermodynamic principles leaving an equation of state to account for the detailed physics of the problem.

We only consider here *isotropic* porous media where the local flow always points in the opposite direction of the pressure gradient. It will be deferred to later to consider more general systems. We also consider for now the pressure gradient as the only driving force in the system.

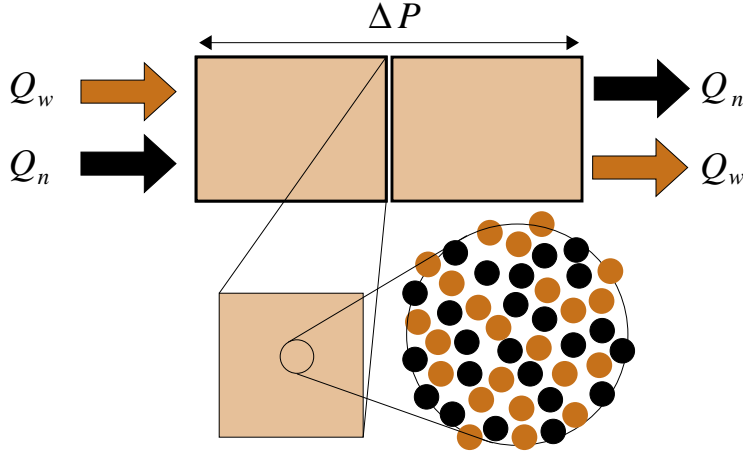
In section 2 we describe the porous medium system we consider. We review the key concepts that will be used in the subsequent discussion. In particular, we discuss the relation between average seepage velocity and the seepage velocities of each of the two fluids. We then ask the central question: given the average seepage velocity and the saturation, is this enough to determine

the seepage velocities of each of the two fluids? The answer seems to be “no,” and we demonstrate why. However, in section 3 we use non-equilibrium thermodynamics to balance dissipation in the two-fluid system against the power supplied to it by the pressure difference across it. We demonstrate that the splitting of the dissipation into that of each fluid is unique. In section 4 we use a scaling argument resting on the Euler theorem for homogeneous functions to set up the framework allowing us to derive a set of equations relating the average seepage velocity, the saturation and the seepage velocities of each of the two fluids. Section 5 contains the derivation of these equations. In section 6 we summarize briefly the theory and point out the role played by the constitutive equation. We derive in section 7 a fractional flow equation, which essentially is a rewriting of one of the two central equations in section 5. Section 8 contains solutions to the two equations of section 5. We obtain integral expressions for the two fluid seepage velocities which are closed by combining them with the constitutive equation. In section 9, we demonstrate the use of the formulas derived in the previous section to relate a given average seepage velocity as function of saturation to the seepage velocities of each of the two fluids. In section 10 we analyze data obtained through a numerical pore scale network model [24]. Section 11 uses the relative permeability equations as constitutive equations demonstrating that the equations derived in section 5 lead to equations between the relative permeabilities and between the capillary pressure and the relative permeabilities. This leads to problems for the relative permeability formulation as the equations predict that at least one of the relative permeabilities must depend on the ratio between the viscosities of the two fluids. We conclude that relative permeability theory is not thermodynamically consistent. In section 12 we write down the full set of equations to describe immiscible two-phase flow in porous media, equations (76) to (79) based on our theory. It reduces the problem from a two-phase fluid problem to a one-phase fluid problem with the saturation as a conserved, extra variable.

## 2 Defining the system

The aim of this paper is to derive a set of equations on the continuum level where differentials make sense from non-equilibrium thermodynamics. We define a *representative elementary volume* — REV — as an isotropic block of porous material filled with two immiscible fluids with no internal structure: It is fully described by a small set of parameters which we will now proceed to define. We will then imagine the volume of the REV shrinking to zero so that the equations we derive become pointwise equations. These equations may then be supplemented by conservation laws to produce a full set of flow equations in the same way as is done with the relative permeability equations.

We show in Fig. 1 the REV. It is a block of homogeneous porous material of length  $L$  and area  $A$ . We seal off the surfaces that are parallel to the  $L$  direction. The two remaining surfaces, each with area  $A$ , are kept open and act as inlet and outlet for the fluids that are injected and extracted from the



**Fig. 1** In the upper part of the figure, we see the REV from the side. A pressure difference  $\Delta P$  is applied across it leading to a flow  $Q = Q_w + Q_n$ . An imaginary cut is made through the REV in the direction orthogonal to the flow. In the lower left corner, the surface of the imaginary cut is illustrated. A magnification of the surface of the cut is shown in the lower right corner. The pore structure is illustrated as brown and black circles. The pores that are brown, are filled with wetting fluid and the pores that are black, are filled with non-wetting fluid. The wetting fluid-filled pores form in total an area  $A_w$  and the non-wetting fluid-filled pores form in total an area  $A_n$ . The total pore area of the imaginary cut in the lower left corner is  $A_p = A_w + A_n$ .

REV. The porosity is  $\phi = V_p/(AL)$  where  $V_p$  is the pore volume.<sup>1</sup> Due to the homogeneity of the porous medium, any cut orthogonal to the axis along the  $L$  direction (named the  $x$  axis for later) will reveal a *pore area* that fluctuates around the value  $A_p = V_p/L = \phi A$ . The homogeneity assumption consists in the fluctuations being so small that they can be ignored.

There is a pressure drop  $\Delta P$  across the REV as shown in Fig. 1. This leads to a time averaged volumetric flow rate  $Q$ . We assume that the REV is isotropic so that  $Q$  — and other volumetric flow rates — are in the direction of the negative pressure gradient. The volumetric flow rate consists of two components,  $Q_w$  and  $Q_n$ , which are the volumetric flow rates of the more wetting ( $w$  for “wetting”) and the less wetting ( $n$  for “non-wetting”) fluids with respect to the porous medium. We have

$$Q = Q_w + Q_n . \quad (1)$$

In the porous medium, there is a volume  $V_w$  of *incompressible* wetting fluid (excluding the irreducible contents of wetting fluid) and a volume  $V_n$  of *incompressible* non-wetting fluid so that  $V_p = V_w + V_n$ . We define the wetting and non-wetting saturations  $S_w = V_w/V_p$  and  $S_n = V_n/V_p$ . We have that

$$S_w + S_n = 1 . \quad (2)$$

<sup>1</sup> We define here that  $V_p$  is the *effective* pore volume that exclude any irreducible wetting fluid or residual non-wetting fluid.

We define the wetting and non-wetting pore areas  $A_w$  and  $A_n$  as the parts of the pore area  $A_p$  which is filled with the wetting or the non-wetting liquids respectively. As the porous medium is homogeneous, we will find the same averages  $A_w$  and  $A_n$  on any cut through the cylindrical porous medium orthogonal to the axis. This is illustrated in Fig. 1. We have that  $A_w/A_p = (A_w L)/(A_p L) = V_w/V_p = S_w$  so that

$$A_w = S_w A_p . \quad (3)$$

Likewise,

$$A_n = S_n A_p . \quad (4)$$

Hence, we have

$$A_p = A_w + A_n . \quad (5)$$

We define the seepage velocities for the two immiscible fluids,  $V_w$  and  $V_n$  as

$$v_w = \frac{Q_w}{A_w} , \quad (6)$$

and

$$v_n = \frac{Q_n}{A_n} . \quad (7)$$

Hence, equation (1) may be written

$$Q = A_w v_w + A_n v_n . \quad (8)$$

We finally define an average seepage velocity associated with the total flow rate  $Q$  as

$$v = \frac{Q}{A_p} . \quad (9)$$

By using equations (3) to (5) and (6), (7) and (9) we transform (8) into

$$v = S_w v_w + S_n v_n . \quad (10)$$

It is important to note that we assume the REV to be small enough so that the saturations and seepage velocities may be taken as constant throughout the sample. There is no internal structure in the REV.

## 2.1 Non-Uniqueness of equation (10)

We pose here the central question that will be answered positively in the next two sections: *from knowledge of  $v$  and  $S_w$  in equation (10), is it possible to determine  $v_w$  and  $v_n$ , defined in equation (6) and (7)?* Seemingly, the answer is “no.” Given a constant reference velocity  $v_0$ , we may define the new velocities

$$\tilde{v}_w = v_w + v_0 S_n , \quad (11)$$

and

$$\tilde{v}_n = v_n - v_0 S_w , \quad (12)$$

and we have

$$A_w \tilde{v}_w + A_n \tilde{v}_n = A_w v_w + A_n v_n = v . \quad (13)$$

Hence, there is seemingly no way to determine which pair of seepage velocities corresponds to equation (6) and (7) without knowing  $Q_w$  or  $Q_n$ . This turns out not to be correct. By balancing the dissipation in the REV against the power delivered to the REV by the action of the pressure difference  $\Delta P$ , we arrive at unique split of the average seepage velocity  $v$  into  $v_w$  and  $v_n$  based on the basic laws of thermodynamics. This is the subject of the next two sections.

### 3 Balancing dissipation and power

The following discussion is based on non-equilibrium thermodynamics [23].

Let  $s = S/V_p$  be the entropy density,  $u = U/V_p$  the internal energy density and  $T$  the temperature of the fluids in the REV. The fluids have chemical potentials  $\mu_w$  and  $\mu_n$ . Their molar concentrations are  $c_w = N_w/V_p$  and  $c_n = N_n/V_p$  where  $N_w$  and  $N_n$  are their respective molar numbers in the REV. The Gibbs relation between these variables is

$$Tds = du - \mu_w dc_w - \mu_n dc_n , \quad (14)$$

This leads to a rate of change of entropy density with time given by

$$\frac{\partial s}{\partial t} = \frac{1}{T} \frac{\partial u}{\partial t} - \frac{\mu_w}{T} \frac{\partial c_w}{\partial t} - \frac{\mu_n}{T} \frac{\partial c_n}{\partial t} , \quad (15)$$

where  $t$  is the time variable. We now introduce conservation equations for entropy, energy and mass of each fluid,

$$\frac{\partial u}{\partial t} + \frac{\partial J_u}{\partial x} = 0 , \quad (16)$$

$$\frac{\partial c_w}{\partial t} + \frac{\partial J_w}{\partial x} = 0 , \quad (17)$$

and

$$\frac{\partial c_n}{\partial t} + \frac{\partial J_n}{\partial x} = 0 , \quad (18)$$

where the  $x$  axis is oriented along the positive flow direction of the REV, see Fig. 1. Here  $J_u$  is the current density of energy. The other two are the current densities of the wetting fluid and non-wetting fluid molar numbers given by

$$J_w = c_w \bar{v}_w , \quad (19)$$

and

$$J_n = c_n \bar{v}_n , \quad (20)$$

where  $\bar{v}_w$  and  $\bar{v}_n$  are the velocities associated with these currents. By combining the conservation laws (16) to (18) with (15) we find

$$\frac{\partial s}{\partial t} = -\frac{\partial}{\partial x} \frac{1}{T} [J_u - J_w \mu_w - J_n \mu_n] + J_u \frac{\partial}{\partial x} \frac{1}{T} - J_w \frac{\partial}{\partial x} \frac{\mu_w}{T} - J_n \frac{\partial}{\partial x} \frac{\mu_n}{T} \quad (21)$$

If we now compare this expression to the entropy balance equation

$$\frac{\partial s}{\partial t} + \frac{\partial J_s}{\partial x} = \sigma , \quad (22)$$

where  $J_s$  is the entropy current density and  $\sigma$  is the entropy production density, we find

$$J_s = \frac{1}{T} [J_u - J_w \mu_w - J_n \mu_n] , \quad (23)$$

and

$$\sigma = J_u \frac{\partial}{\partial x} \frac{1}{T} - J_w \frac{\partial}{\partial x} \frac{\mu_w}{T} - J_n \frac{\partial}{\partial x} \frac{\mu_n}{T} . \quad (24)$$

The system is in a steady state so that  $\partial s / \partial t = 0$ . We also assume that the temperature  $T$  is constant throughout the REV. The steady state assumption also leads to  $\partial J_u / \partial x = \partial J_w / \partial x = \partial J_n / \partial x = 0$  through equations (16) to (18). We integrate the expression for the entropy production (24) over the pore volume of the REV, giving the dissipation  $D$  in the system,

$$D = A_p \int_0^L dx T \sigma = A_p [J_w \Delta \mu_w + J_n \Delta \mu_n] = A_p [c_w \Delta \mu_w \bar{v}_w + c_n \Delta \mu_n \bar{v}_n] , \quad (25)$$

where  $\Delta \mu_w = \mu_w(0) - \mu_w(L)$  and  $\Delta \mu_n = \mu_n(0) - \mu_n(L)$  are the chemical potential differences between the entrance and the exit of the REV for the fluids. The dissipation  $D$  must equal the work done on the system by the pressure difference  $\Delta P$  per time unit,  $Q \Delta P$ . The power supplied to the wetting and the non-wetting fluids by the pressure gradient is  $\bar{Q}_w \Delta P$  and  $\bar{Q}_n \Delta P$  respectively, where  $\bar{Q}_w = V_w \bar{v}_w$  and  $\bar{Q}_n = V_n \bar{v}_n$ . The power supplied to each fluid must balance the dissipation in each fluid if the system is to be in a steady state. Hence,

$$V_w \bar{v}_w \Delta P = V_p c_w \Delta \mu_w \bar{v}_w , \quad (26)$$

and

$$V_n \bar{v}_n \Delta P = V_p c_n \Delta \mu_n \bar{v}_n . \quad (27)$$

Combining these two equations with (25) gives

$$Q \Delta P = A_p v \Delta P = D = A_p [S_w \bar{v}_w + S_n \bar{v}_n] \Delta P , \quad (28)$$

where we have used that  $V_w / V_p = S_w$  and  $V_n / V_p = S_n$ . Hence, we have

$$v = [S_w \bar{v}_w + S_n \bar{v}_n] , \quad (29)$$

in analogy to equation (10). What has been accomplished here is (1) a unique split of the average seepage velocity  $v$  into two velocities  $\bar{v}_w$  and  $\bar{v}_n$  associated with the dissipation in each fluid and (2) we have shown that  $Q$  is a thermodynamic variable.

As we shall see in section 10, in small systems  $\bar{v}_w$  and  $\bar{v}_n$  defined in equations (19) and (20), are not identical to the seepage velocities defined in equations (6) and (7) since they are different averages. However, in the thermodynamic limit which is the continuum limit, they will be equal due to the system being self averaging, and we have

$$\bar{v}_w = v_w = \frac{Q_w}{A_w}, \quad (30)$$

and

$$\bar{v}_n = v_n = \frac{Q_n}{A_n}. \quad (31)$$

#### 4 Scaling assumption

The volumetric flow rate  $Q$ , which we now recognize as a thermodynamic function, is a *homogeneous function of order one* of the two areas  $A_w$  and  $A_n$  defined in equations (3) and (4). Hence, if  $\lambda$  is a scale factor, we may scale the two areas  $A_w \rightarrow \lambda A_w$  and  $A_n \rightarrow \lambda A_n$ . We illustrate this in Fig. 2. This leads to the scaling relation

$$Q(\lambda A_w, \lambda A_n) = \lambda Q(A_w, A_n). \quad (32)$$

This scaling property is essentially self evident. It may be implemented in practice by using different porous media samples with different areas  $A$ .

We take the derivative with respect to  $\lambda$  on both sides of (32) and set  $\lambda = 1$ . This gives

$$Q(A_w, A_n) = A_w \left( \frac{\partial Q}{\partial A_w} \right)_{A_n} + A_n \left( \frac{\partial Q}{\partial A_n} \right)_{A_w}, \quad (33)$$

where it is understood that we keep the pressure drop  $\Delta P$  constant during the partial differentiations. Equation (33) is essentially the Euler theorem for homogeneous functions of order one. By dividing this equation by  $A_p$ , we have

$$v = S_w \left( \frac{\partial Q}{\partial A_w} \right)_{A_n} + S_n \left( \frac{\partial Q}{\partial A_n} \right)_{A_w}. \quad (34)$$

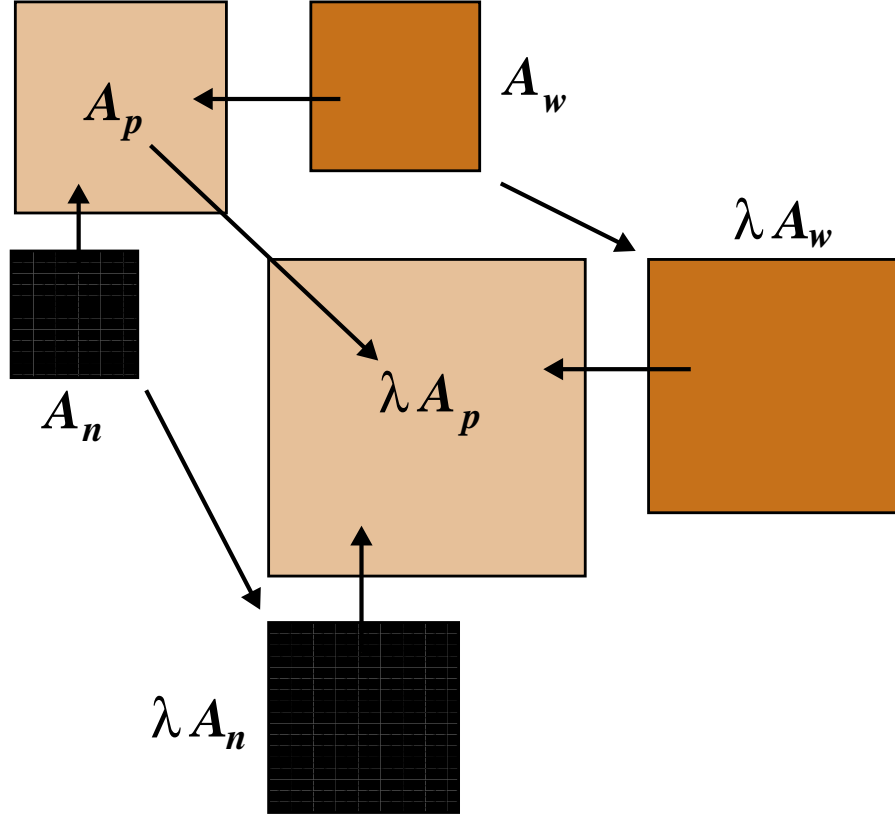
By comparing this equation to (29) using (30) and (31), we find

$$v_w = \frac{Q_w}{A_w} = \left( \frac{\partial Q}{\partial A_w} \right)_{A_n}, \quad (35)$$

and

$$v_n = \frac{Q_n}{A_n} = \left( \frac{\partial Q}{\partial A_n} \right)_{A_w}. \quad (36)$$





**Fig. 2** In the upper left corner, we show an imaginary cut through the REV in the direction orthogonal to the flow as shown in the lower left corner of Fig. 1. The pore area is  $A_p$ . This is decomposed in the area covered by the wetting fluid  $A_w$  and the non-wetting fluid  $A_n$ . These areas are shown below ( $A_w$ ) and to the right ( $A_n$ ) of the cut in the upper left corner. In the center, we show the *rescaled* imaginary cut. That is, we imagine the original REV replaced by another rescaled one. The rescaling has been done with a factor  $\lambda$ . Hence, we have  $A_p \rightarrow \lambda A_p$ , and consequently,  $A_w \rightarrow \lambda A_w$  and  $A_n \rightarrow \lambda A_n$  as show below and to the right of the rescaled cut. The length of the REV,  $L$  is *not* rescaled. is,

## 5 Dependence on saturation: new equations

Let us return to the scaling equation (32) which we combine with the expression for the total flow in terms of the seepage velocities  $v_w$  and  $v_n$ , equation (8). In order for  $Q$  in this equation to obey the scaling relation (32), neither  $v_w$  nor  $v_n$  can be extensive in the variables  $A_w$  and  $A_n$ , i.e.,

$$\begin{aligned}
 v_w(\lambda A_w, \lambda A_n) &= v_w(A_w, A_n) , \\
 v_n(\lambda A_w, \lambda A_n) &= v_n(A_w, A_n) , \\
 v(\lambda A_w, \lambda A_n) &= v(A_w, A_n) .
 \end{aligned}
 \tag{37}$$

They are homogeneous functions of  $A_w$  and  $A_n$  of order zero. Hence, they must depend on the areas  $A_w$  and  $A_n$  through their ratio  $A_w/A_n = S_w/S_n = S_w/(1 - S_w)$ , where we have used equations (3) — (5). Hence,

$$\begin{aligned} v_w &= v_w(S_w) , \\ v_n &= v_n(S_w) , \\ v &= v(S_w) . \end{aligned} \tag{38}$$

The scaling relation (32) implies changing the wetting and non-wetting pore areas  $A_w$  and  $A_n$  by changing  $A_p$ , but without changing the saturations  $S_w = A_w/A_p$  and  $S_n = A_n/A_p$ . We now change the saturation while keeping the total pore area  $A_p = A_w + A_n$  constant. Mathematically, this is accomplished by changing our variables from  $(A_w, A_n)$  to  $(S_w, A_p)$  where  $A_w = S_w A_p$  and  $A_n = (1 - S_w) A_p$ .

We calculate

$$\left( \frac{\partial Q}{\partial S_w} \right)_{A_p} = \left( \frac{\partial A_w}{\partial S_w} \right)_{A_p} \left( \frac{\partial Q}{\partial A_w} \right)_{A_n} + \left( \frac{\partial A_n}{\partial S_w} \right)_{A_p} \left( \frac{\partial Q}{\partial A_n} \right)_{A_w} = A_p [v_w - v_n] , \tag{39}$$

where we have used (35) and (36). We divide by the area  $A_p$  which is kept constant and find

$$\frac{dv}{dS_w} = v_w - v_n . \tag{40}$$

This equation is a direct consequence of scaling assumption (32). This equation is one of the two fundamental equations that constitute the main result presented in this paper.

The second equation we find by taking the derivative of equation (10) with respect to  $S_w$ ,

$$\frac{dv}{dS_w} = \frac{d}{dS_w} [S_w v_w + (1 - S_w) v_n] = v_w - v_n + S_w \frac{dv_w}{dS_w} + (1 - S_w) \frac{dv_n}{dS_w} . \tag{41}$$

Combining this equation with equation (40), we have the equation

$$S_w \frac{dv_w}{dS_w} + (1 - S_w) \frac{dv_n}{dS_w} = 0 , \tag{42}$$

which is our second main result. Also this equation is a consequence of the scaling assumption (32) alone and does not entail any further assumptions about the properties of the flow problem.

Equation (42) is in fact the *Gibbs-Duhem equation*, in disguise, see e.g. Reference [21].

Let us lastly in this section remark that the Darcy or superficial velocities  $V$ ,  $V_w$  and  $V_n$  are related to the seepage velocities by  $V = v\phi$ ,  $V_w = v_w\phi$  and  $V_n = v_n\phi$ . Hence, equations (40) and (42) will not change if expressed in terms of these velocities instead of the seepage velocities.

## 6 Non-equilibrium thermodynamic description

We note that the three equations (10), (40) and (42) are related in such a way that given any two of them, the third follows.

We have accomplished to construct a non-equilibrium thermodynamic theory for immiscible two-phase flow in porous media [22,23]. There are three velocities,  $v$ ,  $v_w$  and  $v_n$ . These are the responses to the pressure difference  $\Delta P$ . Lastly, there is the saturation. The three velocities are bound together by the two equations (40) and (42). Hence, three variables remain: say  $v$ ,  $S_w$  and  $dP/dx = P' = \Delta P/L$ . They will be related through a *constitutive equation*

$$v = v(S_w, P'); \quad (43)$$

It is through this equation that the detailed physics enters the description. We note that this equation plays the same role as an *equation of state* in thermodynamics. The equations we have derived refer only to velocities, saturation and pressure gradient. Hence, the non-zero volume of the REV has dropped out and the two equations may be seen as applying pointwise in a continuous porous medium. If the three equations (40), (42) and (43) are supplied with a conservation law for the saturation, a complete description of porous media in the continuum limit ensues. We elaborate on this in the conclusion and discussion section 12.

## 7 Fractional flow equation

We define the wetting and non-wetting fractional flows as

$$F_w = \frac{Q_w}{Q} = S_w \frac{v_w}{v}, \quad (44)$$

and

$$F_n = 1 - F_w = \frac{Q_n}{Q} = S_n \frac{v_n}{v}, \quad (45)$$

where we have used equations (6) and (7). We now combine these expressions for  $F_w$  and  $F_n$  with equation (40) to find

$$\frac{dv}{dS_w} = v \left[ \frac{F_w}{S_w} - \frac{F_n}{S_n} \right]. \quad (46)$$

By solving for  $F_w$ , we find

$$F_w = S_w + S_w(1 - S_w) \frac{1}{v} \frac{dv}{dS_w}. \quad (47)$$

Hence, we have derived a new *fractional flow equation* relating the fractional flow to the total flow rate and the saturation. Again, no further assumptions have been made concerning the flow apart from the scaling relation given in (32). All that is needed to determine the fractional flow is contained in the average seepage velocity  $v = v(S_w)$ , i.e., in the constitutive law (43).

## 8 Solving the equations

Equations (40) and (42) may be integrated to find  $v_w$  and  $v_n$  as functions of  $S_w$ . We start by transforming equation (40) into

$$S_w \frac{d^2 v}{dS_w^2} = S_w \frac{dv_w}{dS_w} - S_w \frac{dv_n}{dS_w} . \quad (48)$$

Equation (42) may then be subtracted from this equation to give

$$\frac{dv_n}{dS_w} = -S_w \frac{d^2 v}{dS_w^2} . \quad (49)$$

By using equation (42) once again, we find

$$\frac{dv_w}{dS_w} = S_n \frac{d^2 v}{dS_w^2} . \quad (50)$$

We integrate equations (49) and (50) finding

$$v_w(S_w) = v_w(1) - \int_{S_w}^1 dS (1 - S) \frac{d^2 v}{dS^2} , \quad (51)$$

and

$$v_n(S_w) = v_n(0) - \int_0^{S_w} dS S \frac{d^2 v}{dS^2} . \quad (52)$$

As with the fractional flow equation (47), only the average seepage velocity  $v = v(S_w)$  is needed to determine seepage velocities for each of the fluids. This is no surprise as  $F_w$  and  $v_w$  convey the same information about the system.

## 9 Two examples

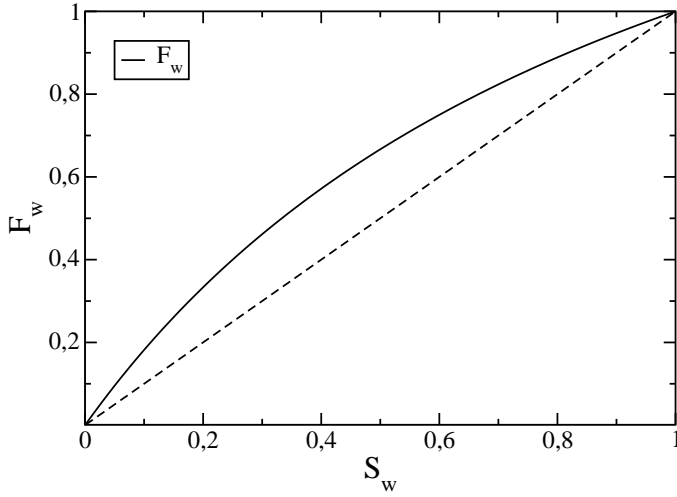
We now give a couple of analytically tractable examples of the use of equations (51) and (52).

First a trivial example. Suppose that surface tension between the two fluids is negligible. The total average seepage velocity (10) is  $v(1) = v_w(1)$  for  $S_w = 1$  and it is  $v(0) = v_n(0)$  for  $S_w = 0$ . If the wetting fluid has a viscosity  $\mu_w$  and the non-wetting fluid a viscosity  $\mu_n$ , we have  $v_w(1)/v_n(0) = \mu_n/\mu_w$ .  $v(S_w)$  must be linear in  $S_w$ :  $v(S_w) = A + BS_w$  and we must have  $A = v_n(0)$  and  $B = v_w(1) - v_n(0)$  so that

$$v(S_w) = v_n(0) + (v_w(1) - v_n(0))S_w = v_n(0) \left[ 1 + \left( \frac{\mu_n}{\mu_w} - 1 \right) S_w \right] . \quad (53)$$

Equations (51) and (52) give since  $d^2 v/dS_w^2 = 0$

$$v_w(S_w) = v_w(1) , \quad (54)$$



**Fig. 3** Fractional flow  $F_w$  as a function of saturation  $S_w$  for the example given in (53) using equation (47). We have set  $\mu_n/\mu_w = 2$ . The broken line is the diagonal  $F_w = S_w$ .

and

$$v_n(S_w) = v_n(0); \quad (55)$$

as it should be.

We may also test the fractional flow equation (47) for this simple case. Using  $v(S_w)$  from (53), we find

$$F_w = S_w + \frac{S_w(1 - S_w)}{1 + (\mu_n/\mu_w - 1)S_w} \left( \frac{\mu_n}{\mu_w} - 1 \right), \quad (56)$$

which is equal to  $S_w v_w / v(S_w)$  as it should be. We illustrate  $F_w$  in Fig. 3.

We know from equation (40) that when  $dv/dS_w = 0$ ,  $v_w = v_n$ . From equation (10) this implies furthermore that  $v = v_w = v_n$ . As a consequence of (47), we also have that  $F_w = S_w$  when  $dv/dS_w = 0$ . Let us name this particular saturation  $S_{w,m}$ . When we move away from  $S_{w,m}$ , we have that

$$v(S_w) = v(S_{w,m}) + \mathcal{O}(\delta S_w^2), \quad (57)$$

where  $\delta S_w = S_w - S_{w,m}$ . Hence, to first order in  $\delta S_w$  the two fluids behave as if they were miscible and both having the same effective viscosity. However, the constitutive equation (43) describing the mixture may still be more complicated than the Darcy equation even if the two fluids each are Newtonian [25, 26, 27, 28, 29].

In the example just discussed, there is no minimum in the average seepage velocity  $v$ , as it is a constant with respect to  $S_w$ . Hence, as long as  $\mu_n \neq \mu_w$ ,  $v_w$  is different from  $v_n$  and — as seen in Fig. 3,  $F_w \neq S_w$  for all values of  $S_w$  except the trivial values 0 and 1.

We now move on to our next example. Let us suppose that the average seepage velocity  $v$  may be parametrized as a fourth order polynomial in  $S_w$ ,

$$v(S_w) = \sum_{j=0}^4 a_j S_w^j. \quad (58)$$

We find from the integrals (51) and (52)

$$v_w(S_w) = (a_0 + a_1) + 2a_2 S_w - (a_2 - 3a_3) S_w^2 - (2a_3 - 4a_4) S_w^3 - 3a_4 S_w^4, \quad (59)$$

and

$$v_n(S_w) = a_0 - a_2 S_w^2 - 2a_3 S_w^3 - 3a_4 S_w^4. \quad (60)$$

A concrete example where  $a_4 = 0$  is

$$v(S_w) = v_n(0) [1 + (\mu_n/\mu_w - 1) S_w - (2 - S_w)(1 - S_w) S_w], \quad (61)$$

see Fig. 4. Integrals (51) and (52) give

$$v_n(S_w) = v_n(0) [1 - S_w^2(3 - 2S_w)], \quad (62)$$

and

$$v_w(S_w) = v_n(0) [\mu_n/\mu_w - 2(1 - S_w)^3]. \quad (63)$$

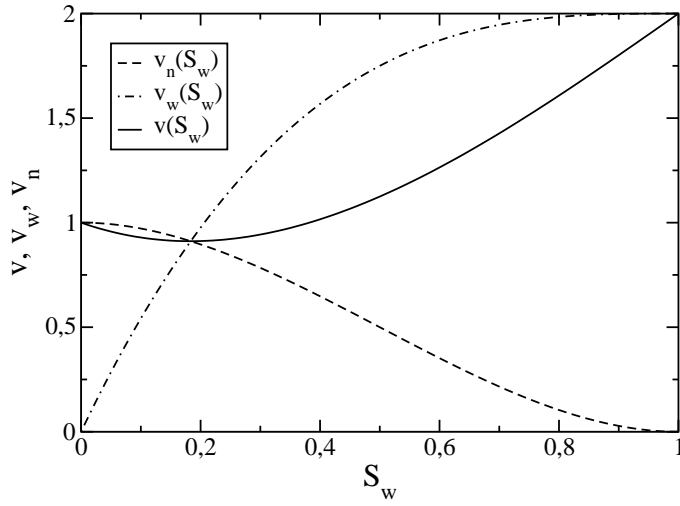
Fig. 4 shows  $v$ ,  $v_w$  and  $v_n$  as function of  $S_w$  for  $\mu_n/\mu_w = 2$ . We note that the three seepage velocities meet as a point. Such a point must exist when  $v_w$  is an increasing function and  $v_n$  is a decreasing function of  $S_w$  as they must cross somewhere, and at the point they meet,  $v$  must be equal to the two others due to equation (10). However, equation (40) dictates that this point is also the point at which  $v$  is minimal. In this case this happens for  $S_{w,m} = 1 - \sqrt{2/3}$ . We see that this is the case in Fig. 4. We show in Fig. 5 the fractional flow  $F_w$  calculated from equation (47) as a function of  $S_w$  for the example given in (61). For the value  $S_{w,m} = 1 - \sqrt{2/3}$ ,  $F_w = S_w$ . As described above, the two immiscible fluids with different viscosities act as if they were miscible and have the same viscosity.

## 10 Numerical studies

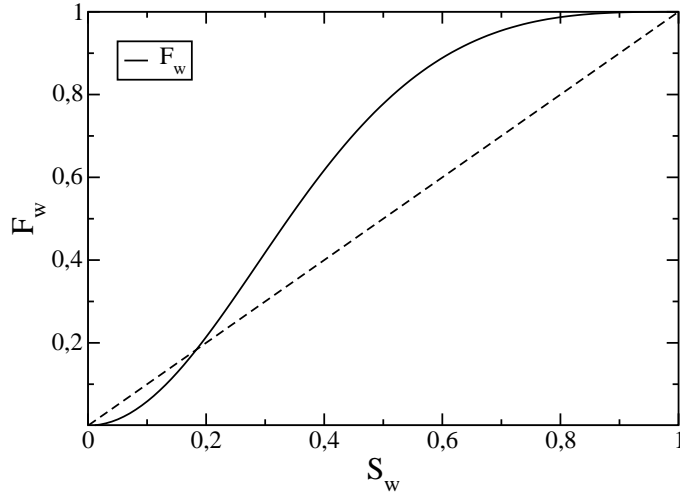
Typically, part of the wetting fluid will be stuck and does not contribute to the flow properties apart from lowering the available pore volume. The volume of this bound wetting fluid divided by the pore volume is the irreducible wetting saturation  $S_{wi}$  and the residual non-wetting saturation  $S_{nrw}$ . It is then convenient to define an effective saturation

$$S_w^* = \frac{S_w - S_{wi}}{1 - S_{wi} - S_{nrw}}. \quad (64)$$

We define the corresponding effective non-wetting saturation as  $S_n^* = 1 - S_w^*$ .



**Fig. 4** Seepage velocities  $v_n$  and  $v_w$  as calculated from the average seepage velocity  $v$  — equation (61) — using equations (51) and (52). We have set  $v_n(0) = 1$  and  $\mu_n/\mu_w = 2$ .



**Fig. 5** Fractional flow  $F_w$  as a function of saturation  $S_w$  for the example given in (61) using equation (47) with  $\mu_n/\mu_w = 2$ . The broken line is the diagonal  $F_w = S_w$ .

The network simulator first proposed by Aker et al. [24] has been refined over the years and is today a versatile model for immiscible two-phase flow under steady-state conditions. The model tracks the interfaces between the immiscible fluids by solving the Kirchhoff equations with a capillary pressure given by

$$p_c = \pm \sum_i^n \frac{2\gamma}{r} (1 - \cos(2\pi x_i)) \quad (65)$$

where  $x_i$  is the position of the  $i$ th interface in a link measured in units of the length  $l$  of the link,  $r$  its average radius and  $\gamma$  is the surface tension between the two fluids. In the following, we will use this model to test the fractional flow equation (47).

Our parameters are chosen as follows: we have set  $l = 1$  mm and  $\gamma = 0.03$  N/m. The link radii  $r$  were drawn from an approximately lognormal distribution with an average of  $0.133l$  and a standard deviation of  $0.044l$ . The viscosities of the fluids were equal,  $\mu_w = \mu_n = 0.1$  Pa s. Steady state is obtained by implementing the network on a torus. Hence, the system is closed and the saturation  $S_w$  does not fluctuate. We have used a two-dimensional hexagonal lattice consisting of  $100 \times 50$  nodes. The system is driven by a constant pressure difference of 15 kPa. The capillary number was hovering around 0.02.

We show in Fig. 6 the volumetric flow rate  $Q$  as a function of  $S_w$  and in Fig. 7 the corresponding fractional flow rate  $F_w$  as a function of  $S_w$ . Our aim is to compare the fractional flow equation (47) to the calculated fractional flow.

In the derivation of the fractional flow equation (47), the velocities defined in equations (30) and (31),  $\bar{v}_w$  and  $\bar{v}_n$  are used. In a network model, one would measure e.g.  $\bar{v}_w$  as follows: let  $q_{w,i}$  be the volumetric flow rate of the wetting fluid in the  $i$ th link and  $a_i$  is the area of this link. Then, the wetting fluid velocity in this link is  $v_{w,i} = q_{w,i}/a_i$ . We then average these velocities over the links. Hence, we have

$$\bar{v}_w = \frac{\sum_i v_{w,i}}{N_L N_A} = \frac{\sum_i \frac{q_{w,i}}{a_i}}{N_L N_A}, \quad (66)$$

where the sum runs over all the links in the network. There are  $N_A$  links in each layer (row of links in the direction orthogonal to the flow direction) and  $N_L$  layers.

The seepage velocity,  $v_w$ , defined in (6) on the other hand, is calculated in the network as

$$v_w = \frac{1}{N_L} \sum_k \left( \frac{\sum_{i \in k} q_{w,i}}{\sum_{j \in k} a_j} \right), \quad (67)$$

where the sum index runs over layers in the network in the flow direction.

In the limit of an infinitely large network the two averages (66) and (67) will be equal as stated in 3. In a small system, such as the one we consider here, they are not,  $v_w \neq \bar{v}_w$ . We measure in Fig. 7  $v_w$  and not  $\bar{v}_w$ .<sup>2</sup>

If we assume that the difference between  $v_w$  and  $\bar{v}_w$  is of the form (11),  $\bar{v}_w = v_w + v_0 S_n$ , we find the following modified fractional flow equation

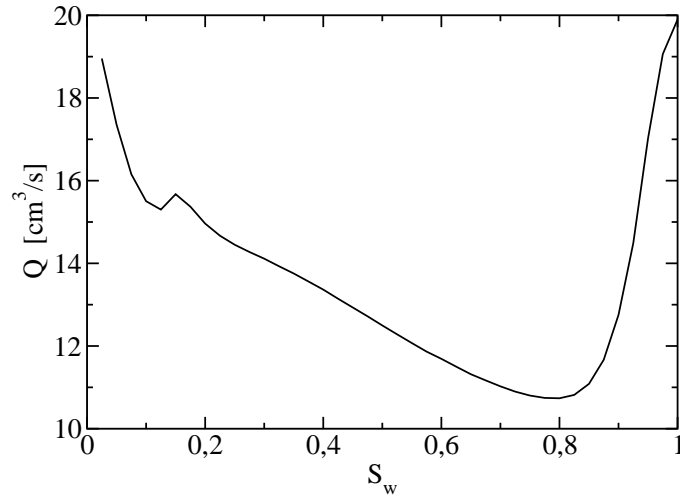
$$F_w = S_w^* + S_w^* (1 - S_w^*) \frac{1}{Q} \left[ \frac{dQ}{dS_w^*} + Q_0 \right]. \quad (68)$$

We have defined  $Q_0 = A_p v_0$ . Furthermore, we have used the effective saturation  $S_w^*$  rather than the saturation  $S_w$ . We have set  $S_{wi} = 0.15$ ,  $S_{nrw} = 0$  and

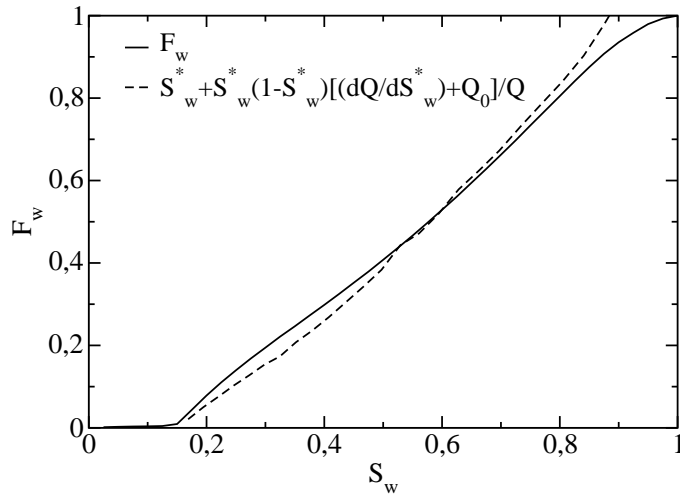
---

<sup>2</sup>  $Q_w$  is measured by keeping track of how much of the wetting fluid has crossed a cross section of the system over a time step.  $F_w$  is then the ratio of how much non-wetting fluid has passed that cross section divided by how much total fluid has passed.





**Fig. 6** Volumetric flow rate  $Q$  as a function of  $S_w$ .



**Fig. 7** Fractional flow rate  $F_w$  and equation (68).

$Q_0 = 4.5 \text{ cm}^3/\text{s}$ . The derivative  $dQ/dS_w$  was calculated by forwards difference with  $\Delta S_w = 0.025$ .

The difference between the measured and calculated fractional flow curves in Fig. 7 is then due to finite-size effects in the averaging process.

## 11 Consequences for relative permeability

Theories of immiscible two-phase flow in porous media attempt to provide a set of equations describing the flow at the continuum scale while anchoring

the key concepts at the pore level. As mentioned in the introduction, there are several different theories in existence. There is the percolation-based approach [5, 19], there is the approach based on continuum mixture theory where the interface energies play a major role [9, 10, 11, 12, 13, 14, 15, 16, 17], and there is the approach where additional variables describing the thermodynamics of the interfaces are introduced [6, 7, 8, 18]. The classical, and by far dominating theory in use today is relative permeability theory [2] even though it is well known to have serious weaknesses.

The two equations (40) and (42) are based on general non-equilibrium thermodynamical arguments that only invoke scale symmetry (32) and a balance of dissipation and power, they must apply to all these theories. In the following, we analyze relative permeability theory in light of our equations.

The relative permeability approach consists in making explicit assumptions about the functional form of  $v_w$  and  $v_n$  through the generalized Darcy equations. Returning to the REV and the definitions of section 2, we have

$$v_w = \frac{1}{\phi S_w} \frac{Q_w}{A} = -\frac{1}{\phi S_w} \frac{K}{\mu_w} k_{r,w} P'_w, \quad (69)$$

and

$$v_n = \frac{1}{\phi S_n} \frac{Q_n}{A} = -\frac{1}{\phi S_n} \frac{K}{\mu_n} k_{r,n} P'_n. \quad (70)$$

Here  $K$  is the permeability of the porous medium, and  $k_{r,w}$  and  $k_{r,n}$  are the relative permeabilities of the wetting and non-wetting fluids. One distinguishes between the pressure gradient in the wetting fluid  $P'_w = \Delta P_w / L$  and in the non-wetting fluid  $P'_n = \Delta P_n / L$ . They are related through the capillary pressure  $P_c$  by

$$P_n - P_w = P_c. \quad (71)$$

The assumptions concerning the physical properties of the flow made in the relative permeability description are that the relative permeabilities and the capillary pressure are functions of the wetting (or equivalently, the non-wetting) saturation *only*,  $k_{r,w} = k_{r,w}(S_w)$ ,  $k_{r,n} = k_{r,n}(S_w)$  and  $P_c = P_c(S_w)$ .

These are strong assumptions, and there is growing evidence that they do not hold in general, see [25, 26, 27, 28, 30, 31, 32]: the relative permeabilities and the capillary pressure are *not* functions of the saturation alone.

Let us now turn to the second fundamental equation (42), combining it with the relative permeability and capillary pressure equations (69)–(71). By inserting the expressions for  $v_w$  and  $v_n$  in terms of the relative permeabilities, and eliminating  $P_w$  by equation (71), we find

$$\begin{aligned} \left[ \frac{S_w}{\mu_w} \left( \frac{d}{dS_w} \right) \left( \frac{k_{r,w}}{S_w} \right) + \frac{1-S_w}{\mu_n} \left( \frac{d}{dS_w} \right) \left( \frac{k_{r,n}}{1-S_w} \right) \right] \Delta P_n \\ - \left[ \frac{S_w}{\mu_w} \left( \frac{d}{dS_w} \right) \left( \frac{k_{r,w}}{S_w} P_c \right) \right] = 0. \end{aligned} \quad (72)$$

This equation must be valid for all values of  $\Delta P_n$ . Hence, it splits into two equations,

$$\frac{S_w}{\mu_w} \left( \frac{d}{dS_w} \right) \left( \frac{k_{r,w}}{S_w} \right) + \frac{1 - S_w}{\mu_n} \left( \frac{d}{dS_w} \right) \left( \frac{k_{r,n}}{1 - S_w} \right) = 0, \quad (73)$$

and

$$\left( \frac{d}{dS_w} \right) \left( \frac{k_{r,w}}{S_w} P_c \right) = 0. \quad (74)$$

This latter equation may in turn be integrated to give

$$P_c = P_0 \frac{S_w}{k_{r,w}}, \quad (75)$$

where  $P_0$  is a reference pressure.

We see here that the fundamental assumptions of relative permeability concept are challenged: equation (73) *contains the viscosities*  $\mu_w$  and  $\mu_n$ . Hence, at least one of the relative permeabilities  $k_{r,w}$  and  $k_{r,n}$  *must* depend of the viscosity ratio  $\mu_n/\mu_w$ .

We also see from equation (75) that the capillary pressure  $P_c$  *cannot become negative*. However, under mixed wetting conditions, it does [33].

There are equations relating the capillary pressure to the relative permeability in the literature such as the Brooks-Corey relation [34] or the van Genuchten equation [35]. These equations all build on the work of Purcell [36] who generalized to pore networks the relation between the capillary pressure and the permeability of a single capillary tube via the Young formula for interfacial tension. Equation (75) is different from these earlier equations in that it does not rely on assumptions concerning the physics of the problem apart from the balance of dissipation in and work on the fluids. We also note that popular parametrizations of the relative permeabilities such as that of Corey [37] and that of Lomeland et al. [38] (“LET”-type) do not obey equation (73). Hence, they are thermodynamically inconsistent, as are the basic assumption that the relative permeabilities  $k_{r,w}$  and  $k_{r,n}$  are functions of the saturation  $S_w$  alone.

## 12 Discussion and conclusion

We have in this paper constructed a theory based on non-equilibrium thermodynamics that reduces the immiscible *two-phase flow* in porous media to a *one-phase* flow problem. Let us now consider a three-dimensional isotropic porous medium. Let  $\mathbf{x}$  be a point somewhere in this porous medium. The theory that we have developed can then be summarized by the following set of equations,

$$\frac{\partial S_w}{\partial t} + \nabla \cdot (S_w \mathbf{v}_w) = 0, \quad (76)$$

$$\mathbf{v} = S_w \mathbf{v}_w + (1 - S_w) \mathbf{v}_n, \quad (77)$$

and

$$S_w \frac{d\mathbf{v}_w}{dS_w} + (1 - S_w) \frac{d\mathbf{v}_n}{dS_w} = 0 . \quad (78)$$

Here (76) is the conservation law for the wetting saturation. This expression becomes the Buckley-Leverett equation [20] if we set  $S_w \mathbf{v}_w = F_w \mathbf{v}$ , see equation (44) and take the incompressibility of the fluids into account. The non-wetting saturation  $S_n$  has been eliminated by using the incompressibility of the two fluids. This implies that  $\nabla \cdot \mathbf{v} = 0$  is build into the equation set. Equation (78) is the three-dimensional version of equation (42). The three-dimensional version of equation (40) is  $d\mathbf{v}/dS_w = \mathbf{v}_w - \mathbf{v}_n$ . This equation follows by taking the derivative of (77) with respect to  $S_w$  and using (78). Hence, this equation is also contained in (76) to (78). It may replace either of the equations (77) or (78). These equations are all conservation laws, the last two express power input equals dissipation. Thus, they transcend the details of the porous medium. Equations (76) to (78) are 7 equations. There are ten variables  $S_w$ ,  $\mathbf{v}$ ,  $\mathbf{v}_w$  and  $\mathbf{v}_n$ . The three equations that close the system of equations are the *constitutive equations*

$$\mathbf{v} = \mathbf{v}[\mathbf{x}, S_w(\mathbf{x}), \nabla P(\mathbf{x})] , \quad (79)$$

containing the detailed physics of the system.

We see that the constitutive equation does not contain the seepage velocities of the immiscible fluids,  $\mathbf{v}_w$  and  $\mathbf{v}_n$ , explicitly. Only the saturation  $S_w$  enters. Hence, the constitutive equation (79) can be interpreted as that of a single fluid depending on one extra variable,  $S_w(\mathbf{x})$ . Hence, the equation set (76) — (79) reduces the immiscible two-phase flow problem in porous media to a one-phase flow problem involving a complex fluid. This viewpoint permeates recent work on the effective permeability of immiscible two-phase systems where it is suggested that two fluids behave as if they were a single Bingham plastic [25, 26, 27, 28, 29].

We may clarify this point even further by eliminating the two fluid velocities  $\mathbf{v}_w$  and  $\mathbf{v}_n$  in equations (76) to (78). The equation set then reduces to a single equation

$$\frac{\partial S_w}{\partial t} + \mathbf{v} \cdot \nabla S_w = S_w \left( \frac{d\mathbf{v}}{dS_w} \right) \cdot \nabla S_w . \quad (80)$$

Together with the constitutive equation (79), we now have a closed set describing effectively a single fluid with a velocity field  $\mathbf{v}$  which transports an active scalar  $S_w$ .

The velocities of the two immiscible fluids may then be found by using the equations

$$\mathbf{v}_w = \mathbf{v} + (1 - S_w) \frac{d\mathbf{v}}{dS_w} , \quad (81)$$

and

$$\mathbf{v}_n = \mathbf{v} - S_w \frac{d\mathbf{v}}{dS_w} , \quad (82)$$

which may be derived from equation (77) and (78).

In this manuscript we have considered a single driving force that induces the flow: the pressure gradient. Other forces such as buoyancy, temperature gradients and chemical driving forces can be incorporated in the non-equilibrium thermodynamics formalism used in section 3. The same is true for the introduction of more immiscible fluids than two. Hence, equations (76) to (78) may be generalized to include these additional complications.

**Acknowledgements** The authors thank Eirik Grude Flekkøy, Knut Jørgen Måløy, Thomas Ramstad, Per Arne Slotte and Marios Valavanides for interesting discussions on this topic. AH, SK and IS thank VISTA, a collaboration between Statoil and the Norwegian Academy of Sciences, for financial support. SS thanks the Norwegian Research Council, NFR and the Beijing Computational Science Research Center CSRC for financial support.

## References

1. J. Bear, Dynamics of fluids in porous media, Dover, Mineola, 1988.
2. R. D. Wyckoff and H. G. Botset, The flow of gas-liquid mixtures through unconsolidated sands, *J. Appl. Phys.* **7**, 325 (1936).
3. L. A. Richards, Capillary conduction of liquids through porous mediums, *J. Appl. Phys.* **1**, 318 (1931).
4. M. C. Leverett, Capillary behavior in porous sands, *Trans. AIMME*, **12**, 152 (1940).
5. R. G. Larson, L. E. Scriven and H. T. Davis, Percolation theory of two phase flow in porous media, *Chem. Eng. Sci.* **36**, 57 (1981).
6. S. M. Hassanizadeh and W. G. Grey, Mechanics and thermodynamics of multiphase flow in porous media including interphase boundaries, *Adv. Wat. Res.* **13**, 169 (1990).
7. S. M. Hassanizadeh and W. G. Grey, Towards an improved description of the physics of two-phase flow, *Adv. Wat. Res.* **16**, 53 (1993).
8. S. M. Hassanizadeh and W. G. Grey, Thermodynamic basis of capillary pressure in porous media, *Wat. Res. Res.* **29**, 3389 (1993).
9. R. Hilfer, Macroscopic equations of motion for two-phase flow in porous media, *Phys. Rev. E*, **58**, 2090 (1998).
10. R. Hilfer and H. Besserer, Macroscopic two-phase flow in porous media, *Physica B*, **279**, 125 (2000).
11. R. Hilfer, Capillary pressure, hysteresis and residual saturation in porous media, *Physica A*, **359**, 119 (2006).
12. R. Hilfer, Macroscopic capillarity and hysteresis for flow in porous media, *Phys. Rev. E*, **73**, 016307 (2006).
13. R. Hilfer, Macroscopic capillarity without a constitutive capillary pressure function, *Physica A*, **371**, 209 (2006).
14. R. Hilfer and F. Döster, Percolation as a basic concept for capillarity, *Transp. Por. Med.* **82**, 507 (2010).
15. J. Niessner, S. Berg and S. M. Hassanizadeh, Comparison of two-phase Darcy's law with a thermodynamically consistent approach, *Transp. Por. Med.* **88**, 133 (2011).
16. F. Döster, O. Hönig and R. Hilfer, Horizontal flow and capillarity-driven redistribution in porous media, *Phys. Rev. E*, **86**, 016317 (2012).
17. R. Hilfer, R. T. Armstrong, S. Berg, A. Georgiadis and H. Ott, Phys. Capillary saturation and desaturation, *Rev. E*, **92**, 063023 (2015).
18. S. M. Hassanizadeh, Advanced theories for two-phase flow in porous media, in *Handbook of Porous Media*, 3rd edition, edited by K. Vafai, CRC Press, Boca Raton, 2015.
19. B. Ghanbarian, M. Sahimi and H. Daigle, Modeling relative permeability of water in soil: Application of effective-medium approximation and percolation theory, *Water Res. Res.* **52**, 5025 (2016).
20. S. E. Buckley and M. C. Leverett, Mechanism of fluid displacements in sands, *Trans. AIME*, **146**, 107 (1942).

21. D. Kondepudi and I. Prigogine, *Modern thermodynamics*, Wiley, Chichester, 1998.
22. S. Kjelstrup and D. Bedeaux, *Non-equilibrium thermodynamics for heterogeneous systems*, World Scientific, Singapore, 2008.
23. S. Kjelstrup, D. Bedeaux, E. Johannesen and J. Gross, *Non-equilibrium thermodynamics for engineers*, World Scientific, Singapore, 2010.
24. E. Aker, K. J. Måløy, A. Hansen and G. G. Batrouni, A two-dimensional network simulator for two-phase flow in porous media, *Transp. Porous Media*, **32**, 163 (1998).
25. K. T. Tallakstad, H. A. Knudsen, T. Ramstad, G. Løvoll, K. J. Måløy, R. Toussaint and E. G. Flekkøy, Steady-state two-phase flow in porous media: statistics and transport properties, *Phys. Rev. Lett.* **102**, 074502 (2009).
26. K. T. Tallakstad, G. Løvoll, H. A. Knudsen, T. Ramstad, E. G. Flekkøy and K. J. Måløy, Steady-state simultaneous two-phase flow in porous media: an experimental study, *Phys. Rev. E* **80**, 036308 (2009).
27. E. M. Rassi, S. L. Codd and J. D. Seymour, Nuclear magnetic resonance characterization of the stationary dynamics of partially saturated media during steady-state infiltration flow, *N. J. Phys.* **13**, 015007 (2011).
28. S. Sinha and A. Hansen, Effective rheology of immiscible two-phase flow in porous media, *Europhys. Lett.* **99**, 44004 (2012).
29. S. Sinha, A. T. Bender, M. Danczyk, K. Keepseagle, C. A. Prather, J. M. Bray, L. W. Thrane, J. D. Seymour, S. L. Codd and A. Hansen, Effective rheology of two-phase flow in three-dimensional porous media: experiment and simulation, submitted to *Trans. Por. Med.* (2016).
30. A. G. Yiotis, L. Talon and D. Salin, Blob population dynamics during immiscible two-phase flow in reconstructed porous media, *Phys. Rev. E*, **87**, 033001 (2013).
31. R. G. Bentsen and J. Trivedi, On the construction of an experimentally based set of equations to describe cocurrent or countercurrent, two-phase flow of immiscible fluids through porous media, *Transp. in Por. Media*, **99**, 251 (2013).
32. M. Moura, E. -A. Florentino, K. J. Måløy, G. Schäfer and R. Toussaint, Impact of sample geometry on the measurement of pressure-saturation curves: experiments and simulations, *Water Res. Res.* **51**, 8900 (2015).
33. W. Abdallah, J. S. Buckley, A. Carnegie, J. Edwards, E. Fordham, A. Graue, T. Habashy, H. Husain, B. Montaron and M. Ziauddin, *Fundamentals of wettability*, *Technology*, **38**, 1125 (1986).
34. R. H. Brooks and A. T. Corey, Hydraulic properties of porous media, Colorado State University hydrology paper **3**, Colorado State University, 1964.
35. M. Th. van Genuchten, A closed-form equation for predicting the hydraulic conductivity of unsaturated soils, *Soil Sci. Soc. Am. J.* **44**, 892 (1980).
36. W. R. Purcell, Capillary pressure — their measurement using mercury and the calculation of permeability therefrom, *Journal Petr. Tech.* **1**, 39 (1949).
37. A. T. Corey, The interrelation between gas and oil relative permeabilities, *Prod. Monthly*, **19**, 38 (1954).
38. F. Lomeland, E. Ebeltoft and T. W. Hammervold, A new versatile relative permeability correlation, *Rev. Proc. of the 2005 Int. Symp. of the SCA, Abu Dhabi, Oct. 31 – Nov. 2, 2005*.